Boolean–Lie algebras and the Leibniz rule

2006 J. Phys. A: Math. Gen. 39 6871

(http://iopscience.iop.org/0305-4470/39/22/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6871–6876 doi:10.1088/0305-4470/39/22/005

Boolean–Lie algebras and the Leibniz rule

Fülöp Bazsó[1,2] **and Elemér Lábos**[3]

[1] KFKI Research Institute for Particle and Nuclear Physics of the Hungarian Academy of Sciences, PO Box 49, H-1525 Budapest, Hungary
[2] Department of Mathematics and Physics, Polytechnical Engineering College, Marka Oreškovića 16, 24000 Subotica, Serbia–Montenegro
[3] Neurobiology Research Group, United Research Organization of the Hungarian Academy of Sciences and Semmelweis University, H-1450 Budapest, PO Box 95, Hungary

E-mail: bazso@sunserv.kfki.hu and labos@ana.sote.hu

Received 3 November 2005, in final form 19 January 2006
Published 16 May 2006
Online at stacks.iop.org/JPhysA/39/6871

Abstract

Using internal negations acting on Boolean functions, the notion of Boolean–Lie algebra is introduced. The underlying Lie product is the Boolean analogue of the Poisson bracket. The structure of a Boolean–Lie algebra is determined; it turns out to be solvable, but not nilpotent. We prove that the adjoint representation of an element of the Boolean–Lie algebra acts as a derivative operator on the space of Boolean functions. The adjoint representation is related to the previously known concept of the sensitivity function. Using the notion of adjoint representation we give the definition of a temporal derivative applicable to iterative dynamics of Boolean mappings.

PACS numbers: 02.10.−v, 02.10.Hh, 45.05.+x

1. Introduction

The success in understanding the behaviour of continuous dynamical systems originates, to a great extent, from the power of infinitesimal calculus. Therefore, it seems reasonable to expect that properly developed calculus can deepen our understanding of iterative dynamics of Boolean functions, discrete dynamical systems and dynamics of computation. There has been a significant advance in geometrization of discrete systems ([7] and references therein) and application of symmetry to special classes of switching gates [4], but no geometric formalism exists for dynamics in the Boolean context, where one operates with truth values and truth functions. To achieve this goal, we clarify the proper mathematical formalism to introduce the basic notions of dynamics and give a correct definition of the derivation in the case of Boolean functions.

The operators used in switching theory, coding and error detection having properties similar to the derivative operators already exist: Boolean derivative [1], sensitivity function

[6] and the total differential [8]. All these operators are linear with respect to the exclusive or, and have desirable properties. Unfortunately, none of these operators obeys the Leibniz rule. The Lie algebras formed of Boolean functions give the appropriate framework in which derivatives emerge in a natural way.

1.1. Notation and basic notions

The following notation is used: Boolean algebra with n atoms is denoted as $\mathcal{B}^n$. $\mathfrak{B}^{m,n}$ denotes the Boolean algebra of $\mathcal{B}^m \to \mathcal{B}^n$ functions, which is isomorphic to $\mathcal{B}^{n2^m}$. Negation of x is denoted as $\overline{x}$, xy is the shorthand notation for the conjunction of x with y, and $\oplus$ denotes the exclusive or. For $x, y \in \mathcal{B}^m$ conjunction and exclusive or are defined componentwise, e.g. $(x_1, \ldots, x_m) \oplus (y_1, \ldots, y_m) = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$. For an explicit denotation of conjunction, the symbol '·' is used. Composition of $\mathfrak{B}^{m,m}$ functions is denoted as $\circ$, i.e. $(F \circ G)(X) \stackrel{\text{def}}{=} F(G(X))$.

The internal negation of a $\mathfrak{B}^{m,1}$ function induced by the index subset $\mathcal{I} = \{i_1, \ldots, i_k\}$, $k \leqslant m$, is defined as $f'_{\mathcal{I}}(x_1, \ldots, x_m) = f(x_1, \ldots, \overline{x}_{i_1}, \ldots, \overline{x}_{i_k}, \ldots, x_m)$, i.e. the variables with the appropriate indices are negated. Let $\langle\mathcal{I}\rangle = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$ denote an n-tuple of index subsets, $F \in \mathfrak{B}^{m,n}$, $F = (f_1, \ldots, f_n)$ and let $F'_{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} (f'_{1\mathcal{I}_1}, \ldots, f'_{n\mathcal{I}_n})$. For any $F \in \mathfrak{B}^{m,n}$, its dual with respect to the internal negation $\langle\mathcal{I}\rangle$ is defined as $\overline{F}'_{\langle\mathcal{I}\rangle}$. The symbols $1_{m,n}, 0_{m,n} \in \mathfrak{B}^{m,n}$ denote the constant mappings identically true and identically false.

The partially self-dual (antidual) functions are defined as $\mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} \{F \in \mathfrak{B}^{m,n} \mid \overline{F}'_{\langle\mathcal{I}\rangle} = F\}$ $\left(\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} \{F \in \mathfrak{B}^{m,n} \mid \overline{F}'_{\langle\mathcal{I}\rangle} = \overline{F}\}\right)$. Antidual functions are invariant under the internal negation. Here, we generalized the notion of an antidual function introduced in [3]. One may easily verify that antidual $\mathcal{A}^{m,1}_{\langle\mathcal{I}\rangle}$ functions form a Boolean algebra with 2^{m-1} atoms. More generally, for any internal negation generated by the index subsets $\langle\mathcal{I}\rangle$, $\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ is isomorphic to $\mathfrak{B}^{m-1,n}$. On the other hand, the set $\mathcal{S}^{m,1}_{\langle\mathcal{I}\rangle}$ forms an antichain of length $2^{(2^{m-1})}$.

The Boolean derivative of $f \in \mathfrak{B}^{m,1}$ is defined as

$$\frac{\partial f(x_1, \ldots, x_m)}{\partial x_i} = f(x_1, \ldots, x_i, \ldots, x_m) \oplus f(x_1, \ldots, \overline{x}_i, \ldots, x_m). \tag{1}$$

Recalling that $\overline{x} \Leftrightarrow x \oplus 1$, the definition of the Boolean derivative resembles the 'finite difference' of f. If the function f does not depend essentially on x_i, then $\partial f/\partial x_i \equiv 0_{m,1}$ [2]. Higher order partial derivatives are defined as $\partial^k f/\partial x_{i_1} \ldots \partial x_{i_k} = \partial/\partial x_{i_1}(\partial^{k-1} f/\partial x_{i_2} \ldots \partial x_{i_k})$, etc. The following identity is true:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}. \tag{2}$$

Applying definition (1) and using the properties of exclusive or, one finds that the Boolean derivative is a nilpotent operator:

$$\frac{\partial^2 f}{\partial x_i^2} \equiv 0_{m,1}. \tag{3}$$

Let $\mathcal{I} = \{i_1, \ldots, i_k\}$; then the sensitivity function of $f \in \mathfrak{B}^{m,1}$ is defined as

$$\sigma_{\mathcal{I}} f \equiv \frac{\sigma^k f}{\sigma x_{i_1} x_{i_2} \ldots x_{i_k}} \stackrel{\text{def}}{=} f'_{\mathcal{I}}(x_1, \ldots, x_m) \oplus f(x_1, \ldots, x_m). \tag{4}$$

As for Boolean derivative operators, the following identities hold for sensitivity functions: $\sigma_{\mathcal{I}_1}(\sigma_{\mathcal{I}_2} f) = \sigma_{\mathcal{I}_2}(\sigma_{\mathcal{I}_1} f)$, $\sigma^2_{\mathcal{I}} f = \sigma_{\mathcal{I}}(\sigma_{\mathcal{I}} f) \equiv 0_{m,1}$. For $F = (f_1, \ldots, f_n) \in \mathfrak{B}^{m,n}$ and $\langle\mathcal{I}\rangle = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$, the sensitivity function of F is defined as $\sigma_{\langle\mathcal{I}\rangle} F = (\sigma_{\mathcal{I}_1} f_1, \ldots, \sigma_{\mathcal{I}_n} f_n)$.

Boolean derivatives and sensitivity functions are not independent notions, but are expressible through each other [5]. For example, for any $f \in \mathfrak{B}^{m,1}$ in the case $\mathcal{I} = \{i\}$, we have $\partial f/\partial x_i \equiv \sigma_{\mathcal{I}} f$.

Let D denote the Boolean derivative or the sensitivity function. For arbitrary $f, g \in \mathfrak{B}^{m,1}$ let u and v denote the derivatives of f and g, respectively: $Df = u, Dg = v$. Then, the following identities hold:

$$D(f \oplus g) = u \oplus v \tag{5}$$

$$D(fg) = ug \oplus fv \oplus uv. \tag{6}$$

1.2. Basic properties of the sensitivity function

Let $F \in \mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle}$ and $G \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. It is simple to check that FG is neither self-dual nor antidual. Based on (6) one has $\sigma_{\langle\mathcal{I}\rangle}(FG) = G$; thus, the operator $\sigma_{\langle\mathcal{I}\rangle}$ performs a surjective $\mathfrak{B}^{m,n} \to \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ mapping. If $\sigma_{\langle\mathcal{I}\rangle} F = F \oplus F'_{\langle\mathcal{I}\rangle} = 0_{m,n}$, adding F to both sides of the last equality one concludes $\ker(\sigma_{\langle\mathcal{I}\rangle}) \subseteq \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. Having $0_{m,n} = F \oplus F$ and knowing that $F \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ the chain of equalities continues as $F \oplus F'_{\langle\mathcal{I}\rangle} = \sigma_{\langle\mathcal{I}\rangle} F$, which implies $\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle} \subseteq \ker(\sigma_{\langle\mathcal{I}\rangle})$, thus $\ker(\sigma_{\langle\mathcal{I}\rangle}) = \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$.

2. Boolean–Lie algebras

The Lie product $[\ ,\ ]_{\langle\mathcal{I}\rangle} : \mathfrak{B}^{m,n} \times \mathfrak{B}^{m,n} \to \mathfrak{B}^{m,n}$ induced with the index subset system $\langle\mathcal{I}\rangle$ is defined as

$$[F, G]_{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} F'_{\langle\mathcal{I}\rangle} G \oplus F G'_{\langle\mathcal{I}\rangle}. \tag{7}$$

One should note the following easily derivable identity:

$$[F, G]_{\langle\mathcal{I}\rangle} = (\sigma_{\langle\mathcal{I}\rangle} F)G \oplus F(\sigma_{\langle\mathcal{I}\rangle} G). \tag{8}$$

For this reason the Lie product can be understood as the Poisson bracket induced by the sensitivity function $\sigma_{\langle\mathcal{I}\rangle}$.

The Lie product has the following properties:

$$[F, F]_{\langle\mathcal{I}\rangle} = 0_{m,n} \tag{9a}$$

$$[F, G]_{\langle\mathcal{I}\rangle} \oplus [G, F]_{\langle\mathcal{I}\rangle} = 0_{m,n} \tag{9b}$$

$$[F \oplus G, H]_{\langle\mathcal{I}\rangle} = [F, H]_{\langle\mathcal{I}\rangle} \oplus [G, H]_{\langle\mathcal{I}\rangle} \tag{9c}$$

$$\forall F, G \in \mathfrak{B}^{m,n}, \qquad [F, G]_{\langle\mathcal{I}\rangle} \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle} \tag{9d}$$

$$\forall F \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}, \qquad [F, G]_{\langle\mathcal{I}\rangle} = F[1_{m,n}, G]_{\langle\mathcal{I}\rangle} = F(G'_{\langle\mathcal{I}\rangle} \oplus G) = F\sigma_{\langle\mathcal{I}\rangle} G \tag{9e}$$

$$\forall F, G \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}, \qquad [F, G]_{\langle\mathcal{I}\rangle} = 0_{m,n} \tag{9f}$$

$$\forall F, G \in \mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle}, \qquad [F, G]_{\langle\mathcal{I}\rangle} = F \oplus G \tag{9g}$$

$$\forall F \in \mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle},\ \forall G \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}, \qquad [F, G]_{\langle\mathcal{I}\rangle} = G. \tag{9h}$$

Using properties of the exclusive or, it is straightforward to show that the $[\ ,\ ]_{\langle\mathcal{I}\rangle}$ Lie product satisfies the Jacobi identity:

$$[F, [G, H]_{\langle\mathcal{I}\rangle}]_{\langle\mathcal{I}\rangle} \oplus [G, [H, F]_{\langle\mathcal{I}\rangle}]_{\langle\mathcal{I}\rangle} \oplus [H, [F, G]_{\langle\mathcal{I}\rangle}]_{\langle\mathcal{I}\rangle} = 0_{m,n}. \tag{10}$$

For any internal negation $\langle\mathcal{I}\rangle$ the two constant functions $1_{m,n}$ and $0_{m,n}$ are always antidual. They form a Boolean algebra isomorphic with $\mathcal{B}^1$ with respect to componentwise conjunction and disjunction. On elements of $\mathcal{B}^1$, one may define addition as exclusive or and multiplication as conjunction and find out that with these operations the resulting structure is isomorphic to the Galois field $GF(2)$. The Galois field with two elements $1_{m,n}$ and $0_{m,n}$ is denoted as $\mathbb{F}_2$. The conjunction of $F \in \mathfrak{B}^{m,n}$ with $1_{m,n}$ $(0_{m,n})$ coincides with scalar multiplication of F with $1 \in \mathcal{B}^1 (0 \in \mathcal{B}^1)$, i.e. $F \cdot 1_{m,n} = (f_1 \cdot 1_{m,1}, \ldots, f_n \cdot 1_{m,1}) = 1 \cdot (f_1, \ldots, f_n)$, and similarily for $0_{m,n}$.

The Jacobi identity and the identities (9*a*), (9*b*) and (9*c*) justify calling the algebra $\mathfrak{b}_{\langle\mathcal{I}\rangle} = (\mathfrak{B}^{m,n}, \oplus, [\ ,\ ]_{\langle\mathcal{I}\rangle}, \cdot, \mathbb{F}_2)$ a Lie algebra. $\mathfrak{a}_{\langle\mathcal{I}\rangle} = \left(\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}, \oplus, [\ ,\ ]_{\langle\mathcal{I}\rangle}, \cdot, \mathbb{F}_2\right)$ is an Abelian ideal of $\mathfrak{b}_{\langle\mathcal{I}\rangle}$. Based on (9*d*) one has $[\mathfrak{B}^{m,n}, \mathfrak{B}^{m,n}]_{\langle\mathcal{I}\rangle} \subseteq \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. Let $F \in \mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle}$ and $G \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. Then $[1_{m,n}, FG]_{\langle\mathcal{I}\rangle} = G$; thus $[\mathfrak{B}^{m,n}, \mathfrak{B}^{m,n}]_{\langle\mathcal{I}\rangle} = \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. From (9*f*) it follows that $\left[\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}, \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}\right]_{\langle\mathcal{I}\rangle} = \{0_{m,n}\}$. One concludes that $\mathfrak{b}_{\langle\mathcal{I}\rangle}$ is solvable. From (9*f*) and the surjectivity of $\sigma_{\langle\mathcal{I}\rangle}$ (section 1.2), one has $\left[\mathfrak{B}^{m,n}, \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}\right]_{\langle\mathcal{I}\rangle} = \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$; thus $\mathfrak{b}_{\langle\mathcal{I}\rangle}$ is not nilpotent. The centre of the algebra is $\{0_{m,n}\}$. For these reasons the algebra $\mathfrak{b}_{\langle\mathcal{I}\rangle}$ is not semisimple.

Because $\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ is an ideal, one can determine the equivalence classes of F under the equivalence relation $\sim_{\langle\mathcal{I}\rangle}$ as $F \sim_{\langle\mathcal{I}\rangle} G|\mathrm{mod}\, \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. The definition of the equivalence class is $F \sim_{\langle\mathcal{I}\rangle} G \Leftrightarrow F \oplus G = H \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$, and for H one has $H = H'_{\langle\mathcal{I}\rangle} = (F \oplus G)'_{\langle\mathcal{I}\rangle} = F \oplus G$, which, using the properties of exclusive or, can be rewritten as $F \sim_{\langle\mathcal{I}\rangle} G \Leftrightarrow \sigma_{\langle\mathcal{I}\rangle} F = \sigma_{\langle\mathcal{I}\rangle} G$, i.e. $[F]^{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} \{G|\sigma_{\langle\mathcal{I}\rangle} F = \sigma_{\langle\mathcal{I}\rangle} G\}$. Proving the fact that $\sim_{\langle\mathcal{I}\rangle}$ is an equivalence relation is straightforward, once one recalls that $\sigma_{\langle\mathcal{I}\rangle}$ vanishes on $\mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$. Addition and the Lie product on equivalence classes are defined as $[F]^{\langle\mathcal{I}\rangle} \oplus [G]^{\langle\mathcal{I}\rangle} \stackrel{\text{def}}{=} [F \oplus G]^{\langle\mathcal{I}\rangle}$ and $[[F]^{\langle\mathcal{I}\rangle}, [G]^{\langle\mathcal{I}\rangle}] \stackrel{\text{def}}{=} [[F, G]_{\langle\mathcal{I}\rangle}]^{\langle\mathcal{I}\rangle}$. From the definition of the Lie product on equivalence classes, it immediately follows that the factor algebra $\mathfrak{b}_{\langle\mathcal{I}\rangle}/\mathfrak{a}_{\langle\mathcal{I}\rangle}$ is isomorphic to $\mathfrak{a}_{\langle\mathcal{I}\rangle}$. From $\sigma_{\langle\mathcal{I}\rangle}([F, G]_{\langle\mathcal{I}\rangle}) = 0_{m,n} = [\sigma_{\langle\mathcal{I}\rangle} F, \sigma_{\langle\mathcal{I}\rangle} G]_{\langle\mathcal{I}\rangle}$, it follows that $\sigma_{\langle\mathcal{I}\rangle}$ is a Lie homomorphism.

The adjoint representation of $F \in \mathfrak{B}^{m,n}$ is defined as usual: $Ad^{\langle\mathcal{I}\rangle}_F \stackrel{\text{def}}{=} [F,\ \ ]_{\langle\mathcal{I}\rangle}$. From the validity of the Jacobi identity, it follows that the adjoint representation acts as a derivative operator on $\mathfrak{B}^{m,n}$ functions. Based on (9*e*) the sensitivity function can be expressed as $\sigma_{\langle\mathcal{I}\rangle} = Ad^{\langle\mathcal{I}\rangle}_{1_{m,n}}$. Finally, one may note that for any $G \in \mathfrak{B}^{m,n}$ one has $Ad^{\langle\mathcal{I}\rangle}_F \left(Ad^{\langle\mathcal{I}\rangle}_F(G)\right) = Ad^{\langle\mathcal{I}\rangle}_F(G)$ if and only if $F \in \mathcal{S}^{m,n}_{\langle\mathcal{I}\rangle}$, i.e. the adjoint operator is idempotent in this case, while for $F \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ the following holds: $Ad^{\langle\mathcal{I}\rangle}_F \left(Ad^{\langle\mathcal{I}\rangle}_F(G)\right) = 0_{m,n}$. The nilpotency of $Ad^{\langle\mathcal{I}\rangle}_F$ for $F \in \mathcal{A}^{m,n}_{\langle\mathcal{I}\rangle}$ points to $\mathfrak{a}_{\langle\mathcal{I}\rangle}$ as a nil ideal of $\mathfrak{b}_{\langle\mathcal{I}\rangle}$ ([9], p 206). In fact, $\mathfrak{a}_{\langle\mathcal{I}\rangle}$ is the nil radical of $\mathfrak{b}_{\langle\mathcal{I}\rangle}$.

3. Temporal derivative

Finally, we give the proper definition of the temporal derivative. For $F \in \mathfrak{B}^{n,n}$ the iterative dynamics is defined as $X_{t+1} = F(X_t)$. Based on the definition of the Boolean derivative (1), the naïve definition of the temporal derivative would be $\mathrm{d}_t F(X) = F(X) \oplus X$, because it compares two consecutive states of the system. This attempt results in a 'derivative' operator which is neither linear with respect to exclusive or, nor obeys the Leibniz rule. One may rewrite the naïve derivative as $(Id \oplus F)(X)$, i.e. an exclusive or of F with the identity function, acting on X. Because Id is a self-dual function, and based on the validity of (9*g*), one would like to 'make' F self-dual. It is possible to construct a self-dual function $\mathcal{F} : \mathcal{B}^{n+1} \to \mathcal{B}^{n+1}$ which coincides with F, when its domain is restricted to $\mathcal{B}^n$. For this purpose, we note that Boolean functions can be expanded into disjunctive normal forms. For simplicity we choose to

Table 1. The truth table of the temporal derivative for a $\mathcal{B}^2$ to $\mathcal{B}^2$ iteration defined by $F(x, y) = (x \cup y, \overline{x}\,\overline{y})$.

x	y	z	f_1	f_2	$f_1 \oplus x$	$f_2 \oplus y$
0	0	0	0	1	0	1
0	0	1	0	1	0	1
0	1	0	0	1	0	0
0	1	1	1	0	1	1
1	0	0	0	1	1	1
1	0	1	1	0	0	0
1	1	0	1	0	0	1
1	1	1	1	0	0	1

expand the Boolean function $f : \mathcal{B}^{n+1} \to \mathcal{B}^1$ with respect to some subset of its variables coded with the index subset $\mathcal{I} = \{i_1, \ldots, i_k\}$ where $k \leqslant n+1$. The partial disjunctive normal form of a function is $f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}) = \cup_{\lambda_1 \in \{0,1\}, \ldots, \lambda_k \in \{0,1\}} f(\lambda_1, \ldots, \lambda_k, x_{k+1}, \ldots, x_{n+1})$, following [10]. Similar expansion holds when we do not expand with respect to the consecutive variables. If $d(\lambda_1 \ldots \lambda_k)$ denotes the decimal value of the binary number $\lambda_1, \ldots, \lambda_k$, we have $A_{d(\lambda_1,\ldots,\lambda_k)}(x_{k+1}, \ldots, x_{n+1}) = f(\lambda_1, \ldots, \lambda_k, x_{k+1}, \ldots, x_{n+1})$, where $d(\lambda_1, \ldots, \lambda_k)$ is an integer ranging from 0 to $2^k - 1$. Thus, the partial disjunctive normal form can be represented as an ordered 2^k-tuple $\langle A_0, \ldots, A_{2^k-1}\rangle$, where all the A's are functions of the remaining, unexpanded variables. The partial disjunctive normal form of $\sigma_{\mathcal{I}} f$ is $\langle (A_0 \oplus A_{2^k-1}), (A_1 \oplus A_{2^k-2}), \ldots, (A_{2^{k-1}-1} \oplus A_{2^{k-1}}), (A_{2^{k-1}} \oplus A_{2^{k-1}-1}), \ldots, (A_0 \oplus A_{2^k-1})\rangle$. For $f \in \mathcal{S}_{\mathcal{I}}^{n+1,1}$, from the definition of self-duality, it follows that $\sigma_{\mathcal{I}} f = 1_{n+1,1}$. Thus, for the partial disjunctive normal form of a self-dual function, we obtain $A_0 = \overline{A}_{2^k-1}$, $A_1 = \overline{A}_{2^k-2}$ and so on. Let $\mathcal{F}(x_1, \ldots, x_n, z) \stackrel{\text{def}}{=} (F(x_1, \ldots, x_n)z \oplus \overline{F}'_{\langle\mathcal{I}\rangle}(x_1, \ldots, x_n)\overline{z}, z)$. Let $X = (x_1, \ldots, x_n, z) \in \mathcal{B}^{n+1}$, and $\pi_n(X) \stackrel{\text{def}}{=} (x_1, \ldots, x_n)$. When $z = 1$, the value of $\mathcal{F}$ projected to $\mathcal{B}^n$ coincides with the value of F, i.e. $\pi_n(\mathcal{F}(x_1, \ldots, x_n, 1)) = F(x_1, \ldots, x_n)$, whilst for $z = 0$ the value of $\mathcal{F}$ projected to $\mathcal{B}^n$ coincides with the value of $\overline{F}'_{\langle\mathcal{I}\rangle}$, the dual function of F. Thus, $\mathcal{F}(x_1, \ldots, x_n, 1)$ corresponds to the original iterative dynamics on $\mathcal{B}^n$, and $\mathcal{F}(x_1, \ldots, x_n, 0)$ corresponds to the naturally adjoined dual dynamics. Based on the $\mathcal{F}$'s partial disjunctive normal form, it is simple to show that $\mathcal{F}$ is self-dual with respect to internal negation over all of its variables. Let now $\langle\mathcal{I}\rangle$ denote internal negation over all the $n+1$ variables of $\mathcal{F}$, and for $X \in \mathcal{B}^{n+1}$ let $\mathcal{Id}(X) = X$. Using the previous notation, the correct definition of the temporal derivative is

$$\mathrm{d}_t \mathcal{F} = [\mathcal{Id}, \mathcal{F}]_{\langle I\rangle} = Ad_{\mathcal{Id}}^{\langle I\rangle}(\mathcal{F}). \tag{11}$$

Defined in this way, the temporal derivative becomes linear with respect to the exclusive or and it obeys the Leibniz rule, because the adjoint representation does.

3.1. Example

We illustrate the notions related to the temporal derivative with the following example. Let $F = (x \cup y, \overline{x}\,\overline{y}) \in \mathfrak{B}^{2,2}$. For the dual function of F, we have $(xy, \overline{x} \cup \overline{y})$. The function $\mathcal{F} = (f_1, f_2, f_3) \in \mathfrak{B}^{3,3}$ is thus $((x \cup y)z \oplus xy\overline{z}, \overline{x}\,\overline{y}z \oplus (\overline{x} \cup \overline{y})\overline{z}, z)$. The truth tables of f_1, f_2 and the temporal derivatives are summarized in table 1.

The disjunctive normal form of $Ad_{\mathcal{Id}}^{\langle I\rangle}(\mathcal{F})$ can be written as $(x\overline{y}\,\overline{z} \cup \overline{x}yz, xyz \cup xy\overline{z} \cup x\overline{y}\,\overline{z} \cup \overline{x}\,\overline{y}z \cup \overline{x}yz \cup \overline{x}\,\overline{y}\,\overline{z}, zxy \cup z\overline{x}y \cup zx\overline{y} \cup z\overline{x}\,\overline{y})$. From the truth table, one may read the

A functions; for instance, in the first component the two nonzero ones are $A_{d(011)} = A_3$ and $A_{d(100)} = A_4$. Inspection of the truth table immediately reveals the fixed point of iteration: $(x, y) = (0, 1)$, the point where $Ad^{\langle I \rangle}_{Id}(\mathcal{F})$ vanishes irrespective of z.

4. Conclusion

Having introduced the proper framework and precise notions characterizing the evolution, one may create the differential-geometric calculus. By looking for further parallels with classical mechanics and continuous dynamical systems, one hopes to gain insight into Boolean dynamics.

Acknowledgments

The authors were supported with OTKA grants T-043456 and T-38140 (FB). The authors would like to thank the two anonymous referees for their valuable comments. FB would like to thank Géza Györgyi and Kornél Szlachányi for useful discussions.

References

[1] Akers S B Jr 1959 *SIAM J.* **7** 487–98
[2] Gavrilov G P and Sapozhenko A A 1977 *Problem Book in Discrete Mathematics* (Moscow: Nauka (in Russian))
[3] Lábos E 1987 *Dynamical Systems* (*Lecture Notes in Economics and Mathematics* vol 287) ed A B Kurzhanski and K Sigmund (Berlin: Springer) pp 189–206
[4] Van Rentergem Y, De Vos A and Storme L 2005 *J. Phys. A: Math. Gen.* **38** 3555–77
[5] Rudeanu S 1974 *Boolean Functions and Equations* (Amsterdam: North-Holland)
[6] Sellers Jr E F 1968 *Error-Detecting Logic for Digital Computers* (New York: McGraw-Hill)
[7] Talasila V, Clemente-Gallardo J and van der Schaft A J 2004 *J. Phys. A: Math. Gen.* **37** 9705–34
[8] Thayse A 1971 *Philips Res. Rep.* **26** 229–46
[9] Varadarajan V S 1984 *Lie Groups, Lie Algebras, and Their Representations* (New York: Springer)
[10] Yablonsky S V and Lupanov O B 1974 *Discrete Mathematics and Mathematical Problems of Cybernetics* ((Moscow: Nauka) (in Russian))